UNIVERSITY OF LIVERPOOL

JANUARY EXAMINATIONS 2011

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: 3 hours

INSTRUCTIONS TO CANDIDATES

Answer all questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

Question 1.

(a) A salt in magnetic field contains N spin ½ ions at temperature T. The magnetic energy levels for each of the ions are –ε and ε.

i) Write an expression for the population of each level in terms of N, ε, and T. **[2]**

ii) Show that the internal energy can be written as

$$U = -N\varepsilon \frac{\exp(\varepsilon / k_B T) - \exp(-\varepsilon / k_B T)}{\exp(\varepsilon / k_B T) + \exp(-\varepsilon / k_B T)} .$$ **[2]**

iii) Find the low and high T limits of U. **[2]**

iv) Sketch the variation of U, and heat capacity C, with T. **[2]**

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Solution

1(a)

(i)

Population for level -ε, $n_1 = N \exp(\varepsilon/k_B T) / Z$

population for level ε, $n_2 = N \exp(-\varepsilon/k_B T) / Z$

$Z = \exp(\varepsilon/k_B T) + \exp(-\varepsilon/k_B T)$ [B2]

(ii)

The energy $U = n_1(-\varepsilon) + n_2\varepsilon$

$= -\varepsilon N \exp(\varepsilon/k_B T) / Z + \varepsilon N \exp(-\varepsilon/k_B T) / Z$

Substituting Z gives

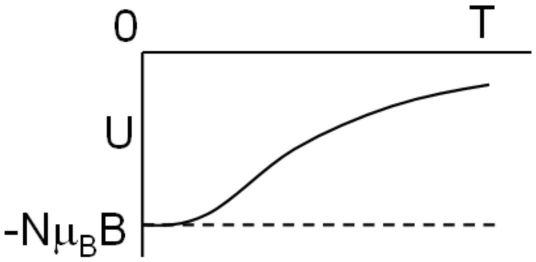
$$U = -N\varepsilon \frac{\exp(\varepsilon / k_B T) - \exp(-\varepsilon / k_B T)}{\exp(\varepsilon / k_B T) + \exp(-\varepsilon / k_B T)}$$ [B2]

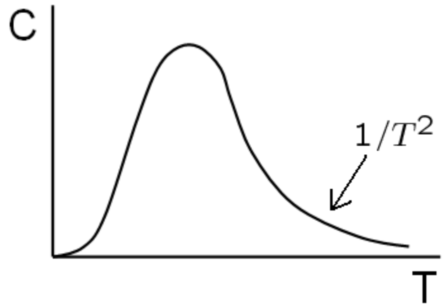
(iii)

$T \to 0$, $\exp(-\varepsilon/k_B T) \to 0$, $U \to -N\varepsilon$

$T \to$ infinity, $\exp(-\varepsilon/k_B T) \to 1$, $U \to 0$ [B2]

(iv)





[B2]

- -

(b) A 1 m^3 box contains one mole of helium at room temperature.

i) Find the average kinetic energy of the atoms. **[2]**

ii) Find the corresponding wavevector. **[2]**

iii) The total number of states with wavevector below k is $G(k) = 4Vk^3/3\pi^2$, where V is the volume. Find the number of states below the average kinetic energy of the atoms. **[2]**

iv) Estimate the probability that one of these states is occupied. **[2]**

v) How does this affect the most probable macrostate? **[2]**

- -

Solution

1(b)

(i) $T = 298$ K.

$E = 3k_BT/2 = 6.172 \times 10^{-21}$ J [U2]

(ii) $m = 4m_u = 6.642 \times 10^{-27}$ kg

$$k = \frac{\sqrt{2mE}}{\hbar} = 8.586 \times 10^{10}\ \text{m}^{-1}.$$ [U2]

(iii) $V = 1\ \text{m}^3$.

$G(k) = 4Vk^3/3\pi^2 = 8.550 \times 10^{31}$ [U2]

(iv) probability = number of atoms / number of states

$\approx N_A / G(k)$

$= 7.043 \times 10^{-9}$ [U2]

(v) This allows us to assume that no two atoms can occupy the same state.

This then leads to the result that the most probable macrostate is the Boltzmann distribution.

[U2]

- -

(c) A system of particles, at temperature T, occupies a set of energy states. The probability that a state at energy ε is occupied, is $f(\varepsilon)$.

i) Write down the expressions for $f(\varepsilon)$ if the particles are fermions, and if they are bosons. [2]

ii) Why are they different? [2]

iii) Sketch a diagram to show how the energy levels of fermions are occupied near 0 K. [2]

iv) Sketch a diagram to show how the energy levels of bosons are occupied near 0 K. [2]

- -

Solution

1(c)

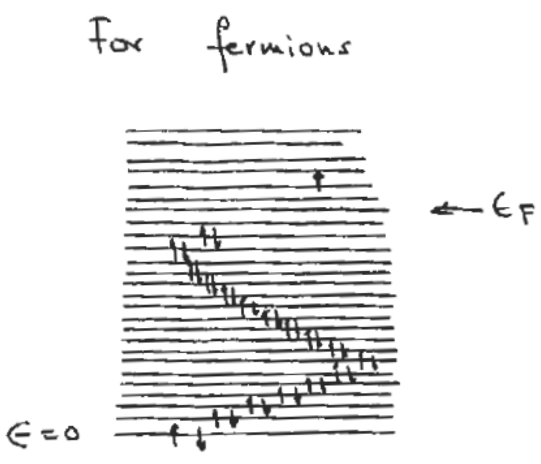
(i)

For fermions, $f(\varepsilon) = 1 / (\exp[(\varepsilon-\mu)/k_BT] + 1)$

For bosons, $f(\varepsilon) = 1 / (\exp[(\varepsilon-\mu)/k_BT] - 1)$ [B2]

(ii) Fermions obey the exclusion principle. Bosons do not. [B2]

(iii)



[B2]

(iv)

For boson

[B2]

- -

(d)

i) The presence of a magnetic field in a macroscopic wavefunction of electrons must produce a current. How does this explain the Meissner's effect? [2]

ii) Discuss how this gives rise to London's penetration depth. [3]

iii) Sketch the heat capacity versus temperature graph for a superconductor, above and below the transition temperature. How does this suggest the existence of an energy gap? [3]

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Solution

1(d)

(i)

When an external field is present, the wavefunction produces a current. This gives a magnetic field that opposes the applied field. So the external field in the superconductor can be cancelled, or expelled. [U2]

(ii)

The current produced is proportional to the charge density of electrons.

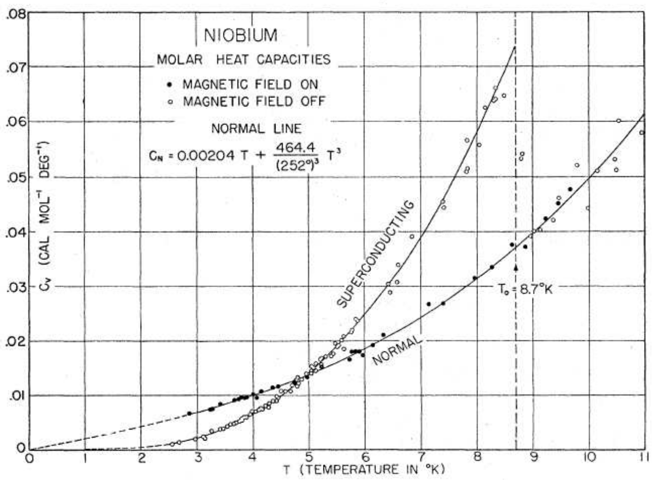
Even if this density is very large, the external field would never be completely expelled.[U1]

The reason is that some current is needed to produce the opposing field.

In order for current to exist, some field must penetrate into the superconductor.

The depth of this penetration is the London's penetration depth. [U2]

iii)



[U1]

Above transition, the heat capacity follows that of the normal metal, $c_v = AT + \gamma T^3$.

Below transition, it changes to $c_v = B\exp(-\Delta/k_BT)$, which has the same form as a Boltzmann factor between two levels of energy gap Δ. [U2]

- -

(e) A body can move with zero resistance through superfluid ^4He.

i) What excitations are possible in superfluid ^4He? [2]

ii) Consider an excitation of energy E and momentum p. The body's velocity must be above E/p before excitation is possible. What are the conservation laws that lead to this? [2]

iii) Sketch the dispersion relation of the excitations in superfluid ^4He. Draw the line with gradient equal to the minimum E/p. [2]

iv) Why does the body experience no resistance when its velocity is below the minimum E/p? [2]

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Solution

1(e)

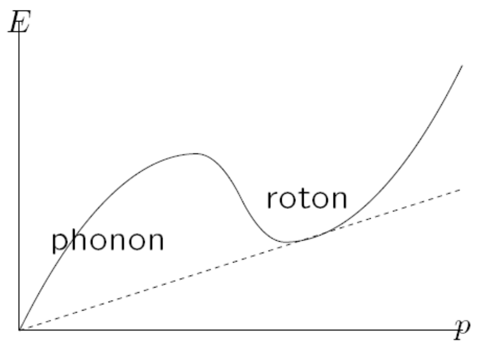
(i) Phonons.

rotons. [B2]

(ii) Conservation of energy

Conservation of momentum [B2]

(iii)



Near p = 0, excitations are due to phonons, which have nonzero gradient.

The gradient of the dashed line is E/p.

It cannot be zero because the roton minimum is above zero. [B2]

(iv)

For velocity below the minimum E/p, no excitation is possible.

If there is no excitation, the body cannot lose energy, so it experiences zero resistance. [B2]

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(f)

i) State the electronic, nuclear and total angular momenta of ^{3}He and ^{4}He atoms. [2]

ii) Explain why one is a fermion and the other is a boson. [2]

iii) The atoms in liquid ^{3}He at 2 K occupy the energy levels in a certain way. Sketch a picture to show this. [2]

iv) The atoms in liquid ^{4}He at 2 K occupy the energy levels in a certain way. Sketch a picture to show this. [2]

- -

Solution

1(f)

(i)

^{3}He angular momentum:

Electronic = 0

Nuclear = ½

Total = ½

^{4}He angular momentum:

Electronic = 0

Nuclear = 0

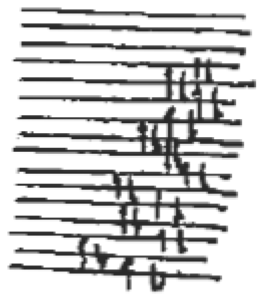
Total = 0 [B2]

(ii)

^3He atom is fermion because its angular momentum is a ½ integer.

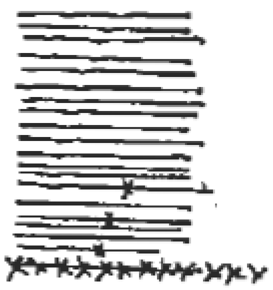
^4He atom is boson because its angular momentum is an integer. [B2]

(iii) ^3He atoms fills up to Fermi energy, with 2 atoms in each energy state:



[B2]

(iv) ^4He atoms mostly condensed to the ground state.



[B2]

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Question 2. Answer **either** (a) **or** (b)

(a) One mole of copper is at 1 K. Each atom supplies two conduction electrons.

i) Assuming that the electrons behave like an ideal gas, write down the expression for the average kinetic energy of the electrons. Hence find the heat capacity. [3]

ii) At 1 K, the measured heat capacity is 0.6 mJ/K. Explain why it is different. [2]

iii) With the help of a graph, estimate the energy range of the electrons that are excited above the Fermi energy at temperature T. [4]

iv) Using the density of states for the ideal gas,

$$g(\varepsilon) = \frac{4m\pi V}{h^3}\sqrt{2m\varepsilon}\ ,$$

derive an expression for the number, n, of excited electrons. (Molar volume of copper is 7.11 cm^3).

[4]

v) Why is it reasonable to suppose that these electrons behave like the ideal gas? [4]

vi) Derive an expression for the heat capacity using the ideal gas assumption. **[4]**

vii) Find the Fermi energy. Calculate the heat capacity for copper at 1 K. Compare with the measured value and comment. [4]

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Solution

2(a)

(i) The average kinetic energy is $\varepsilon = 3k_BT/2$. [U1]

The total energy $U = N_A\varepsilon = N_A \times 3k_BT/2 = 3RT/2$.

The heat capacity $C = dU/dT = 3R/2 = 12.47$ J/K [U2]

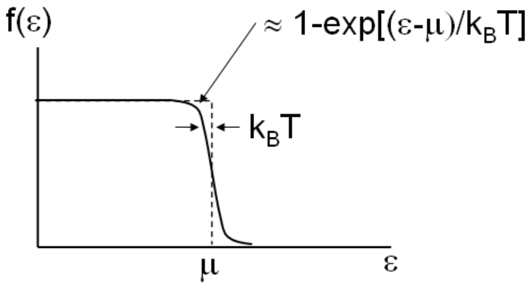
(ii) The electrons do not behave like ideal gas at all.

In the electron gas, the electrons are stacked up in energy levels, from the ground state to a maximum energy. When heated, only a small fraction of the electrons near the top can be excited. As a result, the heat capacity is much smaller. [U2]

(iii) The probability that a state at energy ε is given by the Fermi-Dirac distribution

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_BT)] + 1).$$

At 1 K, the graph is quite close to the step at the Fermi energy $\mu = E_F$.



This is because the Fermi energy for a metal is usually much higher than k_BT. [U2]

So for energy a few k_BT smaller than μ, the exponential function quickly becomes small. Then

$$f(\varepsilon) \approx 1 - \exp[(\varepsilon - \mu)/(k_BT)].$$

When energy falls below μ, the probability moves towards 1 exponentially. It falls roughly to 1/e after k_BT.

So k_BT is approximately the range of energy of the electrons that are excited. [U2]

(iv) The density of states is $2 \times g(\varepsilon) = 2 \times (4m\pi V/h^3)(2m\varepsilon)^{1/2}$.

At the Fermi energy, this is $2g(E_F)$. [U2]

The number of excited electrons is the number of states in their energy range, which is k_BT.

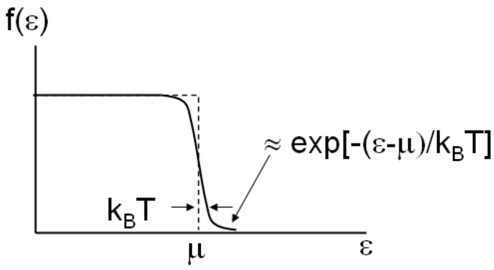
This number n of the states is $2g(E_F) \times k_BT$. [U2]

(v) For energy above E_F, the exponential term in

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_BT)] + 1).$$

quickly gets large, so that

$f(\varepsilon) \approx \exp[-(\varepsilon - \mu)/(k_BT)]$. [U2]



This is just the Boltzmann distribution, the same as that of an ideal gas. [U2]

(vi) The average energy of an ideal gas particle is $3k_BT/2$.

Since the excited electrons behave like an ideal gas, the total energy is $U = n \times 3k_BT/2$. [U2]

Using the previous expression for n, $U = 2g(E_F) \times k_BT \times 3k_BT/2$

$$= 3g(E_F)k_B^2T^2$$

Heat capacity $C = dU/dT = 6g(E_F)k_B^2T$ [U2]

(vii) $N = 2N_A$. The Fermi energy is $E_F = (\hbar^2/2m)(3\pi^2N/V)^{2/3} = 1.789 \times 10^{-18}$ J.

Substituting, we find $C = 6g(E_F)k_B^2T = 0.5777$ mJ/K. [U2]

This is fairly close to the measured 0.6 mJ/K, within 4%.

It is strong evidence that both the Fermi-Dirac distribution, and the ideal gas model of the excited electrons, are correct. [U2]

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(b) N conduction electrons move freely inside a cube of metal of side L.

i) Write quantised values for the wavevector components k_x, k_y, k_z. [2]

ii) Illustrate the allowed states in k space. [2]

iii) Show that the number of states with wavevectors in the range k to k+dk is

$$g(k)dk = \frac{2Vk^2.dk}{\pi^2},$$

where V is the volume L^3. [6]

iv) Write the relation between k and energy ε. [1]

v) Show that the number of states with energy in the range ε to ε+dε is

$$g(\varepsilon) = \frac{2mV}{\pi^2\hbar^3}\sqrt{2m\varepsilon}\,.$$ [3]

vi) The probability that a state with energy ε will be occupied is f(ε). Sketch graphs of f(ε) versus ε for electrons for temperature T = 0 K, and for nonzero T much less that the Fermi temperature T_F. Indicate the Fermi energy E_F. [3]

vii) Show that E_F is given by

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}.$$ [5]

viii) Silver has a molar volume of 10.27 x 10^{-6} m^3. Each atom contributes one conduction electron. Find the Fermi energy. [3]

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Solution

2(b)

(i)

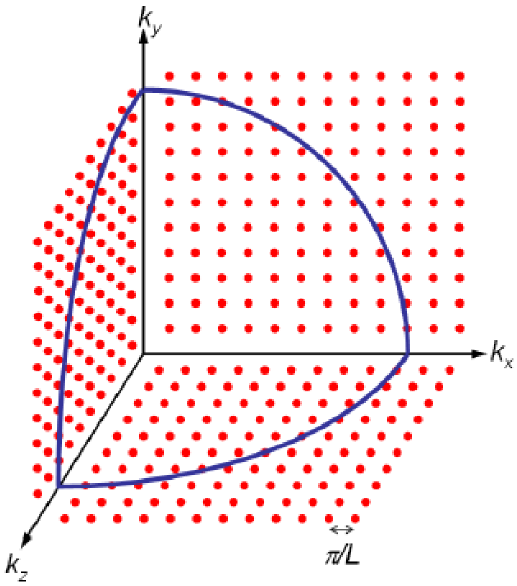
$k_x = n_x\pi/L$

$k_y = n_y\pi/L$

$k_z = n_z\pi/L$

n_x, n_y, n_z are positive integers. [B2]

(ii)



[B2]

(iii)

Number of k states g(k)dk = 2 x volume in k space / volume for each point

Factor 2 is because of the 2 spin states for electrons. [B2]

Volume in k space is a 1/8 of a shell of radius k and thickness dk. So the volume is 1/8 x $4\pi k^2$dk.

Volume for each point is $(\pi/L)^3$. [B2]

Therefore, $g(k)dk = 2 \times 1/8 \times 4\pi k^2 dk / (\pi/L)^3$.

$$g(k)dk = \frac{2Vk^2.dk}{\pi^2}$$

where $V = L^3$. [B2]

(iv)

$$\varepsilon = \frac{\hbar^2 k^2}{2m}$$ [B1]

(v)

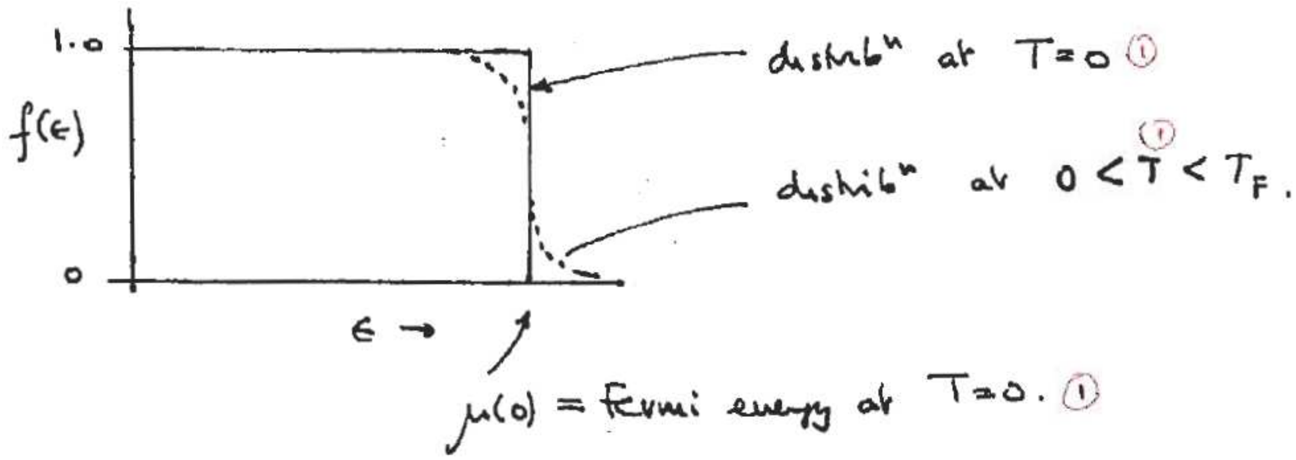
$$k = \frac{\sqrt{2m\varepsilon}}{\hbar} \qquad \frac{d\varepsilon}{dk} = \frac{\hbar^2 k}{m}$$ [B1]

$g(\varepsilon)d\varepsilon = g(k)dk$

g(ε) = g(k) dk/dε

$$g(\varepsilon) = \frac{2Vk^2}{\pi^2}\frac{m}{\hbar^2 k} = \frac{2mVk}{\pi^2\hbar^2} = \frac{2mV}{\pi^2\hbar^2}\frac{\sqrt{2m\varepsilon}}{\hbar} = \frac{2mV}{\pi^2\hbar^3}\sqrt{2m\varepsilon}$$ [B2]

(vi)



[B3]

(vii)

At T = 0, all states are filled up to E_F. [B1]

Let k_F be the corresponding wavevector.

Then $E_F = \dfrac{\hbar^2 k_F{}^2}{2m}$

The number of electrons is

N = 2 x volume of k space / volume for one point

= 2 x (1/8 x $4\pi k_F^3/3$) / $(\pi/L)^3$ [B2]

Rearranging, $k_F^3 = 3\pi^2 N / V$

Substituting into $E_F = \dfrac{\hbar^2 k_F{}^2}{2m}$

we find $E_F = \dfrac{\hbar^2}{2m}\left(\dfrac{3\pi^2 N}{V}\right)^{2/3}$ [B2]

(viii)

N/V = $6.02 \times 10^{23} / 10.27 \times 10^{-6}$ m^3. [B1]

Substituting into the Fermi energy formula, we find $E_F = 9.01 \times 10^{-19}$ J. [B2]

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Question 3. Answer **either** (a) **or** (b)

(a)

i) Why is the Bose Einstein condensate a good candidate for explaining superfluidity and superconductivity? [5]

ii) Using a sketch of the Fermi Dirac distribution graph, explain what happens to the chemical potential μ of a boson gas as temperature falls to 0 K? [5]

iii) In terms of the density of states g(ε), write down the expression for the number of particles N. Explain when and why the number of excited bosons may be written as

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon / k_B T) - 1}. \qquad [5]$$

iv) Given that the solution is

$$N_{ex} = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} 2.612V,$$

explain how to find the condensation temperature T_{BE}. Find the value for liquid ^{4}He, with molar volume 36.84 cm^3. [5]

v) In terms of g(ε), write down the expression for the energy U below T_{BE}. Given that

$$U = 0.7704 k_B N \frac{T^{5/2}}{T_{BE}^{3/2}}$$

is the solution, derive the heat capacity C. Calculate the value of C for one mole of ^{4}He at T_{BE}, and sketch the graph of C versus T. [5]

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Solution

3(a)

(i)

BEC is a macroscopic wavefunction. The number of particles condensed to the ground state is a significant fraction of the bulk total.

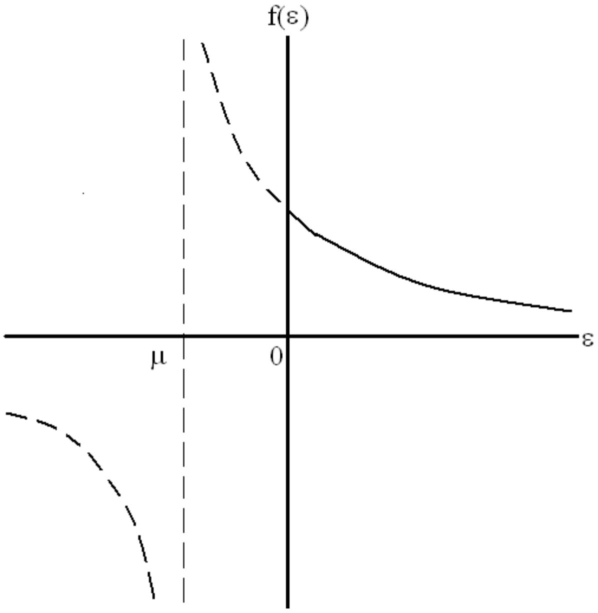
The wavefunction is in one energy state. The next higher state is separated by some amount of energy. If there is not enough energy, the particles cannot get excited. [U2]

Atoms in a superfluid flow round an obstacle without any viscosity. Electrons in a superconductor flow past impurities without any resistance.

This is possible if the atoms or electrons form a macroscopic wavefunction. If the flow is slow, there is not enough energy to excite the particle. Then there is no energy loss, so no resistance.[U2]

Since the BEC is a macroscopic wavefunction, this makes it a good candidate. [U1]

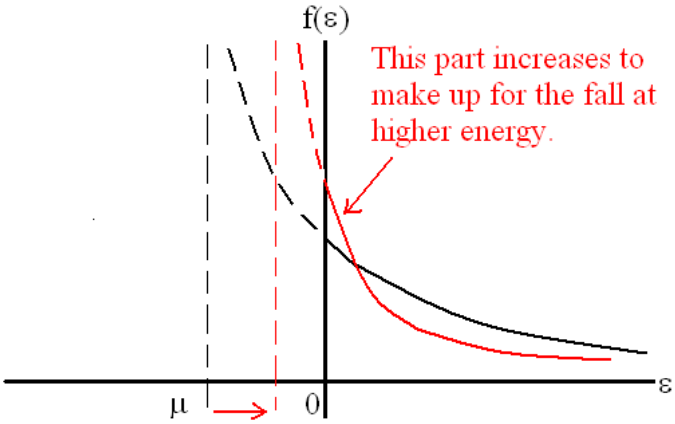
(ii)



Energy is positive. So $\mu < 0$, else $f(\varepsilon)$ can be negative. [U2]

As $T \to 0$, $f(\varepsilon) = 1/[\exp((\varepsilon-\mu)/k_B T)-1]$ falls. [U1]

So that $N = \int g(\varepsilon)f(\varepsilon)d\varepsilon$ remains the same, μ must $\to 0$



[U2]

(iii)

$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon-\mu)/k_B T)-1}$$ [U1]

When T falls, $\mu \to 0$ so that integral remain = N.

When $\mu = 0$, T is defined as T_{BE}.

μ cannot become positive. If T falls below T_{BE}, then integral becomes < N. [U2]

Define $N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon / k_B T) - 1}$.

Interpretation: $N - N_{ex}$ condensed to ground state. So N_{ex} is excited electrons. [U2]

(iv)

At T_{BE}, the particles just start condensing.

So $N_{ex} = N$ the total. [U2]

So can find T_{BE} using

$$N = \left(\frac{2\pi m k_B T_{BE}}{h^2} \right)^{3/2} 2.612V$$ [U1]

Use $N = N_A$ and $V = 36.84$ cm^3, solve for T_{BE}, get 3.13 K. [U2]

(v)

$$U = \int_0^\infty \frac{\varepsilon g(\varepsilon)d\varepsilon}{\exp(\varepsilon / k_B T) - 1}$$ [U1]

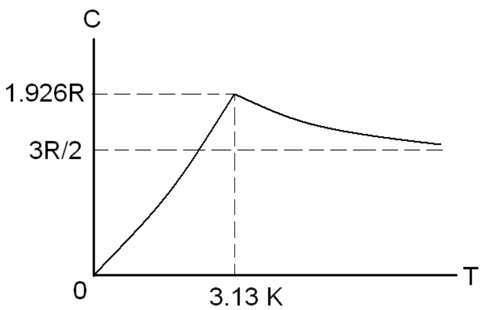
C = dU/dT = $0.7704 k_B N \frac{(5/2)T^{3/2}}{T_{BE}^{3/2}}$

Use $N = N_A$, $T = T_{BE}$.

Find $C = 0.7704 k_B N_A (5/2) = 1.926$ R. [U2]

T → 0, C → 0.

T → large, C → ideal gas result 3R/2.



[U2]

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(b)

i) Describe qualitatively the basic features of the theory of superconductivity. **[5]**

ii) Explain qualitatively what happens as the temperature of a superconductor rises above the critical temperature. **[2]**

iii) Explain qualitatively what happens as a magnetic field higher than the critical field is applied to a superconductor. **[2]**

iv) Describe the isotope effect in superconductivity. What does it prove? **[4]**

v) Describe the Meissner effect. **[5]**

vi) Give two main characteristics of high-T_C superconductor materials. **[2]**

vii) How does the magnetic flux quantum confirm the BCS theory? Mention one important application of magnetic flux quantization. **[5]**

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Solution

a) (i)

Current carriers (electrons in normal conductors) couple in pairs (Cooper pairs) [B1]

Electron spins in a pair are anti-alligned giving a zero-spin object; pairs are bosons [B1]

Pairs have L=0 and S=0 [B1]

Electrons coupling takes place through individual electron coupling to the material lattice [B1]

Pairs do not get excited in collisions, hence no resistance [B1]

Many Cooper pairs occupy ground state forming Bose-Einstein condensate [B1]

(ii)

Average energy kT excites electrons breaking up Cooper pairs [B2]

(iii)

An external field higher than the critical field tends to align electron spins thus breaking up Cooper pairs [B2]

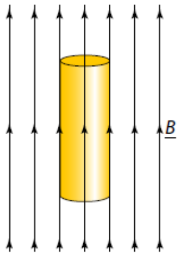
(iv)

The critical temperature values T_C for different isotopes with atomic mass M of the same element are related through: $T_c \propto \sqrt{1/M}$ [B2]

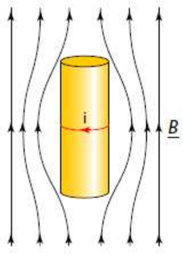
The dependence of T_C on one of the basic lattice parameters proves that Copper pair coupling involves (interactions with) the lattice. [B2]

(v)

Inside magnetic field B:



normal conductor



superconductor [B2]

Material in superconductive state expels all magnetic field lines [B1]

Surface currents generate field (M) opposed to the external one (H), leading to zero field inside (B) [B1]

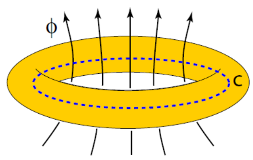
$\vec{B} = \mu_0 \left(\vec{M} + \vec{H} \right) = 0 \Rightarrow \vec{H} = -\vec{M}$ perfect diamagnetism [B1]

(vi)

-. almost insulator at room temperature **[B1]**

- strongly type II when superconducting, flux lines can penetrate **[B1]**

(vii)



Superconductor ring placed in magnetic field at $T>T_C$; then temperature lowered below T_C; supercurrents flowing near the surface of the ring maintain flux constant. [B1]

BCS theory predicts quantization of magnetic flux in the above configuration:

$\Phi = n\phi_0$ with flux quantum: $\phi_0 = \dfrac{h}{q}$ and q the current carrier charge [B2]

Experimental measurements give value of q=2e proving that current in superconductor is carried by electron pairs [B2]

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CONSTANTS

Speed of light in vacuum	c	=	3.00×10^{8} ms^{-1}
Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$ Hm^{-1}
		=	$4\pi \times 10^{-7}$ VsA^{-1}m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12} Fm^{-1}
		=	8.85×10^{-12} AsV^{-1}m^{-1}
Elementary charge	e	=	1.60×10^{-19} C
Planck constant	h	=	6.63×10^{-34} Js
	$h/2\pi = \hbar$	=	1.05×10^{-34} Js
Avogadro constant	N_A	=	6.02×10^{23} mol^{-1}
Boltzmann constant	k_B	=	1.38×10^{-23} JK^{-1}
Gas constant	R	=	8.31 JK^{-1}mol^{-1}
Unified atomic mass constant	m_u	=	1.66×10^{-27} kg
		=	931.5 MeVc^{-2}
Electron mass	m_e	=	9.11×10^{-31} kg
Proton mass	m_p	=	1.67×10^{-27} kg
Gravitational constant	G	=	6.67×10^{-11} Nm^{2}kg^{-2}
Acceleration due to gravity	g	=	9.8 ms^{-2}
Bohr magneton	μ_B	=	9.27×10^{-24} JT^{-1}